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L-S category of quaternionic Stiefel manifolds

Daisuke Kishimoto

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

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Abstract

By calculating certain generalized cohomology theory, lower bounds for the L-S category of quaternionic Stiefel manifolds are given.

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1. Introduction

We consider the *normalized* L-S category throughout, that is the L-S category of a space X, $\mathrm{cat}(X)$, is the least number n such that there exists a cover of X by $(n+1)$ open subsets each of which is contractible in X. Then $\mathrm{cat}(\mathrm{pt}) = 0$.

Let h^* be a multiplicative cohomology theory. The cup-length of a space X with respect to h^*, denoted by $\mathrm{cup}_h(X)$, is the greatest number n such that there exist $x_1, \dots, x_n \in \tilde{h}^*(X)$ with $x_1 \cdots x_n \neq 0$. It is well known that the L-S category is bounded below by the cup-length as

$$\mathrm{cat}(X) \geqslant \mathrm{cup}_h(X) \tag{1}$$

(see, for example, [1]).

Let $X_{n,k}$ denote the quaternionic Stiefel manifold of orthonormal k-frames in $\mathbb{H}^n$, that is $X_{n,k} = Sp(n)/Sp(n-k)$. The purpose of this paper is to give lower bounds for the L-S category of quaternionic Stiefel manifolds $X_{n,k}$. We will employ the same cohomology theory h^* as which was used by Iwase and Mimura [3] to determine $\mathrm{cat}(Sp(3))$. We will investigate the Atiyah–Hirzebruch spectral sequence for h^* and calculate the ring $h^*(X_{n,n-k})$. By this calculation together with (1), we obtain:

Theorem 1.

$$\mathrm{cat}(X_{n,n-k}) \geqslant \begin{cases} n-k & n < 2k+2, \\ n-k+1 & n = 2k+2, 2k+3, \\ n-k+2 & n > 2k+3. \end{cases}$$

E-mail address: kishi@math.kyoto-u.ac.jp (D. Kishimoto).

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From Theorem 1, we can easily deduce the following corollaries concerning the L-S category of $Sp(n)$. Since $Sp(2) = S^3 \cup e^7 \cup e^{10}$, $\mathrm{cat}(Sp(2)) \leqslant 3$ by definition of the L-S category. By setting $(n, n-k) = (2, 0)$ in Theorem 1, one has $\mathrm{cat}(Sp(2)) \geqslant 3$. Then we obtain the following result of Schweizer [7].

Corollary 2. $\mathrm{cat}(Sp(2)) = 3$.

By setting $n - k = 0$ in Theorem 1, one gets the following result of Iwase and Mimura [3].

Corollary 3. $\mathrm{cat}(Sp(n)) \geqslant n + 2$ *for* $n \geqslant 4$.

Remark 4. Iwase and Mimura [3] determined that $\mathrm{cat}(Sp(3)) = 5$ by a hard homotopy calculation. The readers may refer to [1] for the L-S category of Lie groups and homogeneous spaces of Lie groups.

2. Cohomology theory h^*

Let $\hat{\mathbf{S}}$ be the spectrum obtained from the sphere spectrum $\mathbf{S}$ by killing $\pi_n^s(\mathbf{S})$ for $n \geqslant 3$, where $\pi_n^s(\mathbf{X})$ denotes the nth stable homotopy group of a spectrum $\mathbf{X}$. Denote the cohomology theory defined by the spectrum $\hat{\mathbf{S}}$, by h^*. This was introduced by Iwase and Mimura [3] as noted in the previous section. We consider the cohomology theory h^* in a somewhat different point of view.

Since $\hat{\mathbf{S}}$ is a ring spectrum, h^* is multiplicative. It is obvious by definition that

$$h^*(\mathrm{pt}) \cong \mathbb{Z}[\eta]/\big(2\eta, \eta^3\big), \quad |\eta| = -1,$$

where η corresponds to the Hopf element in $\pi_1^s(\mathbf{S})$.

Let X be a finite complex. Recall that the Atiyah–Hirzebruch spectral sequence for $h^*(X)$, denoted by $(E_r(X), d_r)$, is the spectral sequence with

$$E_2^{p,q} \cong H^p\big(X; h^q(\mathrm{pt})\big)$$

converging to $h^*(X)$. We shall investigate the second differential d_2. Let $\mathbf{bo}$ denote the connective KO-spectrum and $({}'E_r(X), {}'d_r)$ the Atiyah–Hirzebruch spectral sequence for the connective KO-theory of X. The spectrum $\hat{\mathbf{S}}$ is constructed also from $\mathbf{bo}$ by killing $\pi_n^s(\mathbf{bo})$ for $n \geqslant 3$. Then there is a map $\mathbf{bo} \to \hat{\mathbf{S}}$ which induces an isomorphism in π_n^s for $0 \leqslant n \leqslant 2$. Hence one has a natural isomorphism

$$E_2^{p,q}(X) \cong {}'E_2^{p,q}(X)$$

for any p and $-2 \leqslant q \leqslant 0$. Thus, for the above isomorphism, one obtains

$$d_2^{p,q} = {}'d_2^{p,q}$$

for any p and $-2 \leqslant q \leqslant 0$. On the other hand, Fujii [2] showed that

$${}'d_2^{p,q} = \begin{cases} Sq^2\pi_2 & q = 0, \\ Sq^2 & q = -1, \end{cases}$$

where π_2 is the modulo 2 reduction. Then we obtain:

Lemma 5. *The second differential of the Atiyah–Hirzebruch spectral sequence for the cohomology theory* h^* *is as follows*:

$$d_2^{p,q} = \begin{cases} Sq^2\pi_2 & q = 0, \\ Sq^2 & q = -1, \\ 0 & q \neq 0, -1. \end{cases}$$

3. Calculation of $h^*(X_{n,k})$

We first calculate $h^*(X_{n,k})$ as an $h^*(\mathrm{pt})$-module by use of the Atiyah–Hirzebruch spectral sequence. Miller [6] gave a stable splitting of quaternionic Stiefel manifold $X_{n,k}$ as

$$X_{n,k} \underset{s}{\simeq} \bigvee_{q=1}^{k} (G_{k,q})^{F_q},$$

where $G_{l,m}$ and X^E denote the quaternionic Grassmaniann $Sp(l)/Sp(m) \times Sp(l-m)$ and the Thom space of a vector bundle $E \to X$ respectively. Then, in order to calculate $h^*(X_{n,k})$, it is sufficient to consider $h^*((G_{k,q})^{F_q})$.

It is well known that the integral cohomology of $G_{l,m}$ is given by

$$H^*(G_{l,m};\mathbb{Z}) = \mathbb{Z}[a_1,\ldots,a_m,b_1,\ldots,b_{l-m}]/(c_1,\ldots,c_l),$$

where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$ and $|a_i| = |b_i| = 4i$. Then, regarding a vector bundle $E \to G_{l,m}$, it follows from the Thom isomorphism theorem that, for any $x, y \in H^*((G_{l,m})^E;\mathbb{Z})$,

$$|x| - |y| \equiv 0\ (4).$$

Hence, by the degree reason, one can see that the Atiyah–Hirzebruch spectral sequence for $h^*((G_{l,m})^E)$ collapses at the E_2-term. Therefore we obtain:

Proposition 6. *The Atiyah–Hirzebruch spectral sequence for $h^*(X_{n,k})$ collapses at the E_2-term.*

Recall that the integral cohomology of $X_{n,n-k}$ is given by

$$H^*(X_{n,n-k};\mathbb{Z}) \cong \bigwedge(e_{4k+3}, e_{4k+7},\ldots,e_{4n-1}), \quad |e_i| = i.$$

Let $(E_r(X), d_r)$ be as in the previous section. Then $E_2(X_{n,n-k})$ is a free $h^*(\mathrm{pt})$-module and hence, by Proposition 6, so is $E_\infty(X_{n,n-k})$. Therefore the extension of $E_\infty(X_{n,n-k})$ to $h^*(X_{n,n-k})$ is trivial and we obtain:

Proposition 7.

$$h^*(X_{n,n-k}) = \Delta_{h^*(\mathrm{pt})}(x_{4k+3}, x_{4k+7},\ldots,x_{4n-1}), \quad |x_i| = i,$$

where $\Delta_R(a_1,a_2,\ldots)$ denotes the simple system generated by $a_1,a_2,\ldots$ over a ring R.

Next we determine the ring structure of $h^*(X_{n,n-k})$ by use of the projective plane of $\Omega X_{n,k}$. Kono and Kozima [4] showed that

$$H_*\big(\Omega Sp(n);\mathbb{Z}\big) = \mathbb{Z}[z'_2, z'_6,\ldots,z'_{4n-2}], \quad |z'_i| = i, \quad Sq^2_*\pi_2 z'_{4i+2} = \pi_2 z'_{4i},$$

where z'_{4i} is inductively defined by $z'_{4i} = (z'_{2i})^2$ and Sq^2_* denotes the dual of Sq^2. Then one has

$$H_*(\Omega X_{n,n-k};\mathbb{Z}) \cong \mathbb{Z}[z_{4k+2},\ldots,z_{4n-2}], \quad Sq^2_*\pi_2 z_{4i+2} = \pi_2 z_{4i}, \tag{2}$$

where $\pi_*(z'_i) = z_i$ for the projection $\pi : \Omega Sp(n) \to \Omega X_{n,n-k}$.

We abbreviately write the element of $E_2(\Omega X_{n,n-k})$ corresponding to $x \in H^*(\Omega X_{n,n-k}; h^*(\mathrm{pt}))$ by the same symbol x. We also abbreviately write the element of $h^*(\Omega X_{n,n-k})$ coming from $y \in E_2(\Omega X_{n,n-k})$ by $[y]$. Then it follows from Lemma 5 and (2) that:

Proposition 8. *The cohomology class $[\eta(z^2_{4k+2})^*] \in h^{8k+3}(\Omega X_{n,n-k})$ is zero for $n \geqslant 2k+2$, where z^* means the Kronecker dual of z.*

Let us recall fundamental facts about projective planes of H-spaces. Let X be an H-space with the multiplication μ. The projective plane of X, P_2X, is defined as the cofibre of the map

$$\theta = \Sigma\mu - \Sigma p_1 - \Sigma p_2 : \Sigma(X \wedge X) \to \Sigma X,$$

where p_i denotes the ith projection and the sums are taken by use of the suspension comultiplication of ΣX. Consider the following exact sequence induced from the cofibration $\Sigma(X \wedge X) \xrightarrow{\theta} \Sigma X \to P_2 X$:

$$\cdots \to \tilde{h}^{*-2}(X) \xrightarrow{\theta^*} \tilde{h}^{*-2}(X \wedge X) \xrightarrow{\lambda} \tilde{h}^*(P_2 X) \xrightarrow{i} \tilde{h}^{*-1}(X) \to \cdots. \tag{3}$$

It is known that the above exact sequence satisfies that if $i(a) = x$ and $i(b) = y$ then $\lambda(x \times y) = ab$ (see [5]).

Now let us consider the projective plane of $\Omega X_{n,n-k}$. The bottom cells of $\Sigma\Omega X_{n,n-k}$, $P_2\Omega X_{n,n-k}$ and $B\Omega X_{n,n-k} \simeq X_{n,n-k}$ are common by the following sequence of inclusions.

$$S^{4k+3} \hookrightarrow \Sigma\Omega X_{n,n-k} \overset{\bar{i}}{\hookrightarrow} P_2\Omega X_{n,n-k} \overset{j}{\hookrightarrow} B\Omega X_{n,n-k} \simeq X_{n,n-k}.$$

Note that the inclusion $\bar{i}$ induces $i : \tilde{h}^*(P_2\Omega X_{n,n-k}) \to \tilde{h}^{*-1}(\Omega X_{n,n-k})$ in the exact sequence (3). Then, by the degree reason, one can see that

$$i \circ j^*(x_{4k+3}) = [z^*_{4k+2}].$$

Therefore one has

$$\lambda\big(\eta[z^*_{4k+2}] \times [z^*_{4k+2}]\big) = j^*\big(\eta x^2_{4k+3}\big).$$

If there exists $x \in h^*(\Omega X_{n,n-k})$ such that $\theta^*(x) = \eta[z^*_{4k+2}] \times [z^*_{4k+2}]$, then $x = [\eta(z^2_{4k+2})^*]$ by the degree reason. It follows from Proposition 8 that $x = 0$. Then $j^*(\eta x^2_{4k+3}) \neq 0$ and hence

$$x^2_{4k+3} \neq 0$$

when $n \geqslant 2k + 2$. Thus, by Proposition 7 and the degree reason, we have

$$x^2_{4k+3} = \eta x_{8k+7}$$

when $n \geqslant 2k + 2$. Therefore, by considering the projection $X_{n,l} \to X_{n,m}$ for $l > m$, we obtain:

Theorem 9. *For $x_{4l+3}, x_{8l+7} \in h^*(X_{n,n-k})$, we have*

$$x^2_{4l+3} = \begin{cases} \eta x_{8l+7} & n \geqslant 2l + 2, \\ 0 & n < 2l + 2. \end{cases}$$

4. Proof of Theorem 1

For $n < 2k + 2$, $h^*(X_{n,n-k})$ is an exterior algebra by Theorem 9 and hence one has

$$\mathrm{cup}_h(X_{n,n-k}) = n - k.$$

For $n = 2k + 2, 2k + 3$, it follows from Theorem 9 that

$$x^3_{4k+3} x_{4k+7} \cdots x_{8k+3} x_{8k+11} \cdots x_{4n-1} = \eta x_{4k+3} x_{4k+7} \cdots x_{4n-1} \neq 0$$

and hence we have

$$\mathrm{cup}_h(X_{n,n-k}) \geqslant n - k + 1.$$

For $n > 2k + 3$, by Theorem 9, we have

$$x^3_{4k+3} x^2_{8k+7} x_{4k+7} \cdots x_{8k+3} x_{8k+11} \cdots x_{16k+11} x_{16k+19} \cdots x_{4n-1} = \eta^2 x_{4k+3} x_{4k+7} \cdots x_{4n-1} \neq 0.$$

Then we obtain

$$\mathrm{cup}_h(X_{n,n-k}) \geqslant n - k + 2$$

and (1) completes the proof of Theorem 1.

References

[1] O. Cornea, G. Lupton, J. Oprea, D. Tanré, Lusternik–Schnirelmann Category, Mathematical Surveys and Monographs, vol. 103, 2003.
[2] M. Fujii, K_O-groups of projective spaces, Osaka J. Math. 4 (1967) 141–149.
[3] N. Iwase, M. Mimura, L-S categories of simply-connected compact simple Lie groups of low rank, in: Progr. in Math., vol. 215, 2004, pp. 199–212.
[4] A. Kono, K. Kozima, The space of loops in a symplectic group, Japan. J. Math. 4 (2) (1978) 461–486.
[5] J.P. Lin, H-spaces with finiteness conditions, in: Handbook of Algebraic Topology, 1995, pp. 1095–1141.
[6] H. Miller, Stable splittings of Stiefel manifolds, Topology 24 (1985) 411–419.
[7] P.A. Schweizer, Secondary cohomology operations induced by the diagonal mapping, Topology 3 (1965) 337–355.